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# ON STABILITY OF CONVECTIVE FLOW OF A BINARY MIXTURE WITH THERMAL DIFFUSION*

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The linear stability of a steady plane parallel convective flow of a binary mixture in a plane vertical layer is considered with allowance for the thermal diffusion effect. Various instability mechanisms are discussed, and stability boundaries and properties of critical perturbations are determined.

**1. Statement of the problem.** A viscous nonreacting binary mixture is contained in a vertical infinitely long plane layer with boundaries $x = \pm h$ maintained at constant but different temperatures $\mp\Theta$. The boundaries are assumed impermeable to matter. Allowance is made for the effect of thermal diffusion which induces additional flow of matter, proportional to the temperature gradient. Owing to thermal diffusion, the temperature difference of boundaries creates a horizontal concentration gradient. The inhomogeneity of density produced by the temperature and concentration gradients induces a convective flow in the layer.

We formulate the equations of binary mixture convection in the Boussinesq approximation, taking into account thermal diffusion but neglecting the effect of the thermal conductivity diffusion effect. Besides conventional notation we shall use the following: $h$ for length, $h^2/\nu$ for time, $g\beta_1\Theta h^2/\nu$ for velocity, $\Theta$ for temperature, $\beta_1\Theta/\beta_2$ for concentration, and $\rho g\beta_1\Theta h$ for pressure, with $\beta_1$, $\beta_2$ as the temperature and concentration density coefficients. All parameters of the mixture (except density) are assumed independent of temperature and concentration. The dimensionless equations and boundary conditions for velocity $\mathbf{v}$, temperature $T$, light component concentration $C$, and pressure $p$ are of the form

$$\frac{\partial \mathbf{v}}{\partial t} + G(\mathbf{v}\nabla)\mathbf{v} = -\nabla p + \nabla\mathbf{v} + (T + C)\gamma, \quad \nabla\mathbf{v} = 0 \tag{1.1}$$

$$\frac{\partial T}{\partial t} + G\mathbf{v}\nabla T = \frac{1}{P}\Delta T$$

$$\frac{\partial C}{\partial t} + G\mathbf{v}\nabla C = \frac{1}{P_d}(\Delta C - \varepsilon\Delta T); \quad \int_{-1}^{1} v_z\,dx = 0$$

$$x = \pm 1;\quad \mathbf{v} = 0,\quad T = \mp 1,\quad -\frac{\partial C}{\partial x} + \varepsilon\frac{\partial T}{\partial x} = 0, G = \frac{g\beta_1\Theta h^3}{\nu^2},\quad P = \frac{\nu}{\chi},\quad P_d = \frac{\nu}{D},\quad \varepsilon = -\frac{\alpha\beta_2}{\beta_1}$$

The following dimensionless parameters appear in the problem: the Grashof number $G$, the Prandtl number $P$, the Schmidt number $P_d$, and the dimensionless thermal diffusion parameter $\varepsilon$ ($\alpha$ is the thermal diffusion coefficient; for regular and anomalous thermal diffusion $\varepsilon > 0$ and $\varepsilon < 0$), respectively).

Problem (1.1) has a solution that defines a steady plane parallel flow with a cubic velocity profile, in the direction along vertical $z$ axis, and linear temperature and concentration profiles

$$v_0 = \frac{1}{6}(1 + \varepsilon)(x^3 - x), \quad T_0 = -x, \quad C_0 = -\varepsilon x \tag{1.2}$$

In regular thermal diffusion $(\varepsilon > 0)$, when there is a surplus of the light component near the heated wall, velocities of the convection counter-flows increase, unlike in the case of a homogeneous fluid; when this effect is anomalous $(\varepsilon < 0)$, these velocities decrease, and in the case of $\varepsilon = -1$ we have mechanical equilibrium.

To investigate stability of the basic mode (1.2) we consider small normal plane perturbations proportional to $\exp(-\lambda t + ikz)$. For the amplitude perturbations of the stream function $\varphi$, temperature $\theta$, and concentration $\xi$ we have the spectral boundary value problem

$$\begin{aligned} &-\lambda\Delta\varphi + ikG(v_0\Delta\varphi - v_0''\varphi) = \Delta^2\varphi + \theta' + \xi' \\ &-\lambda\theta + ikG(v_0\theta + \varphi) = P^{-1}\Delta\theta \quad (\Delta = d^2/dx^2 - k^2) \\ &-\lambda\xi + ikG(v_0\xi + \varepsilon\varphi) = P_d^{-1}(\Delta\xi - \varepsilon\Delta\theta) \\ &x = \pm 1;\quad \varphi = \varphi' = 0,\quad \theta = 0,\quad \xi' - \varepsilon\theta' = 0 \end{aligned} \tag{1.3}$$

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where primes denote differentation with respect to $x$.

The characteristic decrements $\lambda = \lambda_r + i\lambda_i$ defined by this problem depend on parameters $G$, $P$, $P_d$, $\varepsilon$ and $k$. Stability limits are obtained from the condition that $\lambda_r = 0$.

Problem (1.3) was solved numerically using the method of differential run through /1/, varying the Prandtl numbers $P$ and $P_d$. Neutral curves $G(k)$, whose position depends on the remaining parameters of the problem, were obtained. The stability limit is determined by the minimal value of $G_m(P, P_d, \varepsilon)$ with respect to $k$. Characteristics of critical perturbations, i.e. the wave number $k_m$ and (in the case of wave perturbations) the phase velocity in units of the basic flow maximum velocity $c = 9\sqrt{3}\,\lambda_{im} / [(1+\varepsilon)\,k_m G_m]$ were obtained besides the stability limit.

**2. Discussion of results.** Calculations had shown that in the investigated range of parameters meaning, instability is induced by one of the following three mechanisms. The first is of hydrodynamic nature and associated with the development of vortices at the boundaries of counterflowing convective streams; this instability is of a steady character, with zero phase velocity of respective perturbations. The second mechanism is associated with the development of increasing wave perturbations in ascending and descending streams. Finally, a branch of steady thermal concentration instability induced by long-wave perturbations was revealed in the region of anomalous thermal diffusion.

The combined pattern of flow stability is shown in Fig.1, which represents the dependence of the minimal critical Grashof number $G_m$ on the dimensionless thermal diffusion parameter $\varepsilon$ for the indicated above critical perturbation types. The pattern relates to $P = 6.7$, which is characteristic for liquid mixtures.

Curve 1 corresponds to the stability limit of the hydrodynamic type. When $\varepsilon = 0$ and the stated boundary conditions are satisfied, the gradient of concentration is absent, and from the stability viewpoint the mixture behaves as a homogeneous medium. The critical Grashof number $G_{m0}$ then weakly depends on $P$ (when $P = 0.7$ and 6.7, $G_{m0} = 503$, and 492, respectively) /2/. In conformity with (1.2) variation of $\varepsilon$ results in the amplitude of velocity variation of the basic flow according to the law $(1+\varepsilon)$, which lowers the stability limit in the region of $\varepsilon > 0$ and increases it in that of $\varepsilon < 0$ (of anomalous thermal diffusion), reaching absolute stability as $\varepsilon \to -1$. Within the accuracy of the diagram the stability limit of the hydrodynamic mode in the investigated region of parameters is defined by formula $G_m = G_{m0} / (1+\varepsilon)$, in which the absence of implicit dependence on $P$ and $P_d$ is due to the purely hydrodynamic origin of the respective instability mechanism.

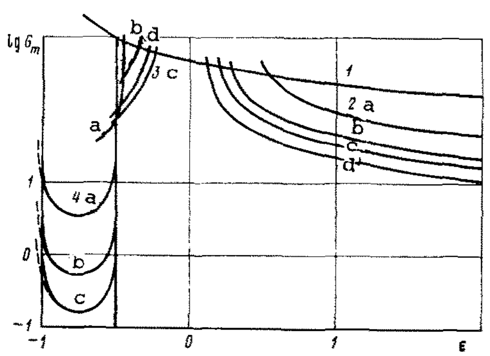


Fig.1

Curves 2a—2d define stability limits relative to wave perturbations and correspond to parameter $P_d = 30$; 100; 200, and 676.7. When $P = 6.7$ and $P_d$ values as indicated, the waves are of the concentration nature. It will be seen that in the wide interval of $\varepsilon > 0$ the wave mode is the more dangerous. As $P_d$ increases the Grashof number $G_m$ decreases; there is an analogy here with dependence $G_m(P)$ for the temperature wave mode in a homogeneous fluid /2/.

There is also a range of values of parameter $\varepsilon$ in which the most dangerous are concentration waves in the case of anomalous thermal diffusion ($\varepsilon < 0$). The respective stability limits are represented by the set of curves 3 (a and b $P_d = 30$, c $P_d = 200$, and d $P_d = 676.7$). The nonmonotonicity of function $G_m(P_d)$ should be noted. Moreover, when $P_d = 30$, there are two intersecting modes (curves a and b) generated by different branches of the concentration perturbation spectrum.

Let us pass to the third instability mechanism indicated above, i.e. of the thermal concentration type. That mechanism acts in the region of considerable thermal diffusion anomaly, and is linked with the development of long-wave perturbations when $k_m = 0$. Stability boundaries 4,a, b, c correspond to $P_d = 30, 200$, and 676.7. In the case of high $P_d$, characteristic of liquid mixtures, the critical Grashof number is low, and a strong destabilization of the flow takes place.

The appearance in the region of irregular thermal diffusion of the thermal concentration instability mechanism is natural. In the case of large absolute values of $\varepsilon$ the temperature and concentration gradients are directed so that partial, or even total (when $\varepsilon = -1$) compensation of respective horizontal density gradients occurs. Boundary conditions that specify a zero stream of matter on the channel walls imply the generation in this situation of long-wave instability by analogy with the flow of a mixture in a vertical layer with a stabilizing longitudinal concentration gradient /3,4/. The long-wave properties of this mode enable us to determine analytically the stability limit using the method of small parameter.

Let us consider the limit case of long-wave perturbations as $k \to 0$. The spectral problem (1.3) with $k = 0$ has evidently a neutral level that corresponds to a uniform amplitude of concentration perturbation over the cross section

$$\lambda = 0, \ \varphi = \theta = 0, \ \xi = \text{const} \tag{2.1}$$

where the normalization constant is assumed equal unity.

We seek a solution whose structure corresponds that level for small $k$, and represent it in the form of expansions

$$\begin{gathered} \varphi = \varphi_1 k + \varphi_2 k^2 + \dots, \ \theta = \theta_1 k + \theta_2 k^2 + \dots \\ \xi = 1 + \xi_1 k + \xi_2 k^2 + \dots, \ \lambda = \lambda_1 k + \lambda_2 k^2 + \dots \end{gathered} \tag{2.2}$$

We write down the systems of first and second order equations

$$\begin{gathered} \varphi_1^{\text{IV}} + \theta_1' + \xi_1' = 0, \ \theta_1'' = 0 \\ \xi_1'' - \varepsilon\theta_1'' = -\lambda_1 P_d + iGP_d v_0 \end{gathered} \tag{2.3}$$

$$\begin{gathered} \varphi_2^{\text{IV}} + \theta_2' + \xi_2' = -\lambda_1\varphi_1'' + iG(v_0\varphi_1'' - v_0''\varphi_1) \\ \theta_2'' = -\lambda_1 P\theta_1 + iGP(v_0\theta_1 + \varphi_1) \\ \xi_2'' - \varepsilon\theta_2'' = 1 - \lambda_1 P_d \xi_1 - \lambda_2 P_d + iGP_d(v_0\xi_1 + \varepsilon\varphi_1) \end{gathered} \tag{2.4}$$

Boundary conditions for all approximations are the same as for problem (1.3).

The condition of solvability of the inhomogeneous system (2.3) yields $\lambda_1 = 0$, and the solution is of the form

$$\varphi_1 = -\frac{iGP_d(1+\varepsilon)}{120960}(1-x^2)^2(163 - 22x^2 + 3x^4)$$

$$\theta_1 = 0, \quad \xi_1 = \frac{iGP_d(1+\varepsilon)}{360}(15x - 10x^3 + 3x^5)$$

The condition of solvability of system (2.4) — vanishing of the integral in the right-hand side of Eqs. (2.4) — yields

$$\lambda_2 P_d = 1 + \frac{2}{2835}(GP_d)^2(1+\varepsilon)(1+2\varepsilon)$$

Since at the stability limit $\lambda_2 = 0$, hence the critical Grashof number

$$G_c P_d = \frac{9}{2}\left[-\frac{70}{(1+\varepsilon)(1+2\varepsilon)}\right]^{1/2} \tag{2.5}$$

which shows that the product $G_c P_d$ dependent only on the thermal diffusion parameter, is the critical parameter. Instability exists in the interval $-1 < \varepsilon < -1/2$. Function (2.5) is in complete agreement with the results of numerical calculations (curves *4* in Fig.1). Moreover, calculations show that almost throughout the indicated interval of values of $\varepsilon$ the minimal critical Grashof number is reached when $k_m = 0$. Nevertheless, when $\varepsilon < -0.98$, the minimum of stability curves $G(k)$ shifts to perturbations with $k_m \neq 0$, which shows that, as in the case of lengthwise stratification /4/, the cell type perturbations are the most dangerous. The respective stability limits are shown in Fig.1 by dash lines.

The dependence of critical perturbation parameters, i.e. the wave-number $k_m$ and the phase velocity $c_r$ on $\varepsilon$ is shown in Figs.2 and 3, where the numbering of curves is the same as in Fig.1. It will be seen that in the region of $\varepsilon > 0$ the phase velocity of critical wave perturbations is close to the maximum velocity of the basic flow, while in that of $\varepsilon < 0$ critical perturbations "lag behind" the latter.

Liquid mixtures can substantially differ with respect to the Prandtl number. The effect of parameter $P$ on the wave instability limit are shown in Fig.4 for fixed $P_d = 676.7$. Curves *1—6* correspond there to the following values of the thermal diffusion parameter: $\varepsilon = 0; 0.1; 0.3; -0.1; -0.3; -0.5$. This inducates that the thermal diffusion leads to a lowering of stability with respect to wave perturbations. The critical Grashof number diminishes with increasing $P$. Depending on $P$ and $P_d$ the wave instability mode is either of the thermal, concentration, or mixed nature.

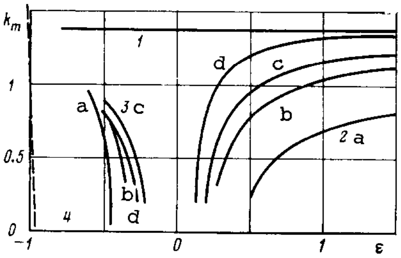


Fig.2

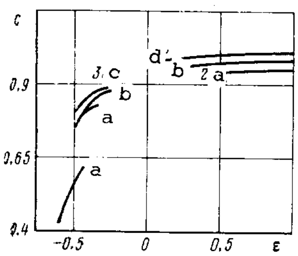


Fig.3

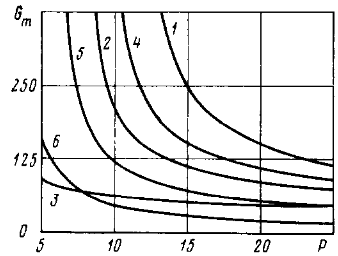


Fig.4

In the case of gas mixtures $P \sim P_d \sim 1$. For these values of parameters the temperature, as well as the concentration wave instability modes are absent. In the diagram of $G_m(\varepsilon)$ corresponding to Fig.1 only curves *1* and *4* remain with the limit of hydrodynamic type instability (curve *1*) depending weakly on $P$ and $P_d$, as previously indicated, while the limit of thermal concentration long-wave type instability (curve *4*) defined by formula (2.5) are entirely independent of $P$. Thus the region of regular and moderately anomalous thermal diffusion, the instability of gas mixture is always due the effect of the hydrodynamic mechanism, while in the region of considerable anomaly of thermal diffusion $(\varepsilon < -1/2)$, the most dangerous is the thermal concentration mechanism.

So far, only the stability with respect to plane perturbations was considered here. Investigation of three-dimensional regular perturbations of the form $\exp[i(k_1y + k_2z)]$ reduces to the spectral problem for respective amplitudes. It can be shown that the problem can always be reduced by some transformations to a similar problem for plane perturbations. The critical Grashof number $G$ in the case of three-dimensional perturbations with wavenumbers $k_1$ and $k_2$ is then expressed in terms of the critical number $G'$ for plane perturbations with wavenumber $k' = (k_1^2 + k_2^2)^{1/2}$ using formula $G = G'/a$, in which $a = k_2/(k_1^2 + k_2^2)^{1/2}$ is the parameter of three-dimensional perturbations varying within the limits $0 \leqslant a \leqslant 1$. Thus $G \geqslant G'$ which shows that the most dangerous are plane perturbations (an analog of Squire's theorem).

The authors are only aware of paper /5/ in which stability of a mixture convective flow was investigated with allowance for the thermal diffusion effect. The flow considered there had a longitudinal concentration gradient, and its stability limit was established only with respect to perturbations of the hydrodynamic type.

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